

Sequential edge-coloring on the subset of vertices of almost regular graphs

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January 7, 2014

Abstract

Let G be a graph and $R \subseteq V(G)$. A proper edge-coloring of a graph G with colors $1, \dots, t$ is called an R -sequential t -coloring if the edges incident to each vertex $v \in R$ are colored by the colors $1, \dots, d_G(v)$, where $d_G(v)$ is the degree of the vertex v in G . In this note, we show that if G is a graph with $\Delta(G) - \delta(G) \leq 1$ and $\chi'(G) = \Delta(G) = r$ ($r \geq 3$), then G has an R -sequential r -coloring with $|R| \geq \left\lceil \frac{(r-1)n_r + n}{r} \right\rceil$, where $n = |V(G)|$ and $n_r = |\{v \in V(G) : d_G(v) = r\}|$. As a corollary, we obtain the following result: if G is a graph with $\Delta(G) - \delta(G) \leq 1$ and $\chi'(G) = \Delta(G) = r$ ($r \geq 3$), then $\Sigma'(G) \leq \left\lfloor \frac{2n_r(2r-1) + n(r-1)(r^2+2r-2)}{4r} \right\rfloor$, where $\Sigma'(G)$ is the edge-chromatic sum of G .

1 Introduction

In this note we consider graphs which are finite, undirected, and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G , respectively. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$ and the chromatic index of G by $\chi'(G)$. For a graph G , let $\Delta(G)$ and $\delta(G)$ denote the maximum and minimum degrees of vertices in G , respectively. An (a, b) -biregular bipartite graph G is a bipartite graph G with the vertices in one part all having degree a and the vertices in the other part all having degree b . The terms and concepts that we do not define can be found in [5].

A proper edge-coloring of a graph G is a mapping $\alpha : E(G) \rightarrow \mathbf{N}$ such that $\alpha(e) \neq \alpha(e')$ for every pair of adjacent edges $e, e' \in E(G)$. If α is a proper

edge-coloring of a graph G , then $\Sigma'(G, \alpha)$ denotes the sum of the colors of the edges of G . For a graph G , define the edge-chromatic sum $\Sigma'(G)$ as follows: $\Sigma'(G) = \min_{\alpha} \Sigma'(G, \alpha)$, where minimum is taken among all possible proper edge-colorings of G . A proper t -coloring is a proper edge-coloring which makes use of t different colors. If α is a proper t -coloring of G and $v \in V(G)$, then $S(v, \alpha)$ denotes set of colors appearing on edges incident to v . Let G be a graph and $R \subseteq V(G)$. A proper edge-coloring of a graph G with colors $1, \dots, t$ is called an R -sequential t -coloring if the edges incident to each vertex $v \in R$ are colored by the colors $1, \dots, d_G(v)$.

The concept of sequential edge-coloring of graphs was introduced by Asratian [1]. In [1, 2], he proved the following result.

Theorem 1. If $G = (X \cup Y, E)$ is a bipartite graph with $d_G(x) \geq d_G(y)$ for every $xy \in E(G)$, where $x \in X$ and $y \in Y$, then G has an X -sequential $\Delta(G)$ -coloring.

On the other hand, in [2] Asratian and Kamalian showed that the problem of deciding whether a bipartite graph $G = (X \cup Y, E)$ with $\Delta(G) = 3$ has an X -sequential 3-coloring is NP -complete. Some other results on sequential edge-colorings of graphs were obtained in [3, 4]. In particular, in [4] Kamalian proved the following result.

Theorem 2. If G is an $(r-1, r)$ -biregular ($r \geq 3$) bipartite graph with n vertices, then G has an R -sequential r -coloring with $|R| \geq \left\lceil \frac{rn}{2r-1} \right\rceil$.

In this note we generalize last theorem. As a corollary, we also obtain the following result: if G is a graph with $\Delta(G) - \delta(G) \leq 1$ and $\chi'(G) = \Delta(G) = r$ ($r \geq 3$), then $\Sigma'(G) \leq \left\lfloor \frac{2n_r(2r-1) + n(r-1)(r^2+2r-2)}{4r} \right\rfloor$.

2 The Result

Theorem 3. If G a graph with $\Delta(G) - \delta(G) \leq 1$ and $\chi'(G) = \Delta(G) = r$ ($r \geq 3$), then G has an R -sequential r -coloring with $|R| \geq \left\lceil \frac{(r-1)n_r + n}{r} \right\rceil$, where $n = |V(G)|$ and $n_r = |\{v \in V(G) : d_G(v) = r\}|$.

Proof. Since $\chi'(G) = \Delta(G) = r$, there exists a proper r -coloring α of the graph G with colors $1, 2, \dots, r$. For $i = 1, 2, \dots, r$, define the set $V_{\alpha}(i)$ as follows:

$$V_{\alpha}(i) = \{v \in V(G) : i \notin S(v, \alpha)\}.$$

Clearly, for any $i', i'', 1 \leq i' < i'' \leq r$, we have

$$V_{\alpha}(i') \cap V_{\alpha}(i'') = \emptyset \quad \text{and} \quad \bigcup_{i=1}^r V_{\alpha}(i) = V(G) \setminus V_r,$$

where $V_r = \{v \in V(G) : d_G(v) = r\}$.

Hence,

$$n - n_r = |V(G)| - |V_r| = \left| \bigcup_{i=1}^r V_\alpha(i) \right| = \sum_{i=1}^r |V_\alpha(i)|.$$

This implies that there exists i_0 , $1 \leq i_0 \leq r$, for which $|V_\alpha(i_0)| \geq \lceil \frac{n-n_r}{r} \rceil$. Let $R = V_r \cup V_\alpha(i_0)$. Clearly, $|R| \geq n_r + \lceil \frac{n-n_r}{r} \rceil$.

If $i_0 = r$, then α is an R -sequential r -coloring of G ; otherwise define an edge-coloring β as follows: for any $e \in E(G)$, let

$$\beta(e) = \begin{cases} \alpha(e), & \text{if } \alpha(e) \neq i_0, r, \\ i_0, & \text{if } \alpha(e) = r, \\ r, & \text{if } \alpha(e) = i_0. \end{cases}$$

It is easy to see that β is an R -sequential r -coloring of G with $|R| \geq \lceil \frac{(r-1)n_r+n}{r} \rceil$. \square

Corollary 1. If G is an $(r-1, r)$ -biregular ($r \geq 3$) bipartite graph with n vertices, then G has an R -sequential r -coloring with $|R| \geq \lceil \frac{rn}{2r-1} \rceil$.

Corollary 2. If G is a graph with $\Delta(G) - \delta(G) \leq 1$ and $\chi'(G) = \Delta(G) = r$ ($r \geq 3$), then

$$\Sigma'(G) \leq \left\lfloor \frac{2n_r(2r-1) + n(r-1)(r^2+2r-2)}{4r} \right\rfloor$$

Proof. Let α be an R -sequential r -coloring of G with $|R| \geq \lceil \frac{(r-1)n_r+n}{r} \rceil$ described in the proof of Theorem 3. Now, we have

$$\begin{aligned} \Sigma'(G) \leq \Sigma'(G, \alpha) &\leq \frac{\frac{n_r \cdot r(r+1)}{2} + \lceil \frac{n-n_r}{r} \rceil \frac{r(r-1)}{2} + (n - n_r - \lceil \frac{n-n_r}{r} \rceil) \frac{(r+2)(r-1)}{2}}{2} \\ &\leq \frac{\frac{n_r \cdot r(r+1)}{2} + \frac{(n-n_r)r(r-1)}{2r} + (n - n_r - \frac{n-n_r}{r}) \frac{(r+2)(r-1)}{2}}{2} \\ &= \frac{\frac{n_r \cdot r(r+1)}{2} + \frac{(n-n_r)r(r-1)}{2r} + \frac{(n-n_r)(r+2)(r-1)^2}{2r}}{2} \\ &= \frac{n_r \cdot r(r+1)}{4} + \frac{(n-n_r)(r-1)(r^2+2r-2)}{4r} \\ &= \frac{2n_r(2r-1) + n(r-1)(r^2+2r-2)}{4r}. \end{aligned}$$

\square

References

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